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Weak similarities of metric and semimetric spaces

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Abstract

Let (X, d_X) and (Y, d_Y) be semimetric spaces with distance sets $D(X)$ and, respectively, $D(Y)$. A mapping $F : X \to Y$ is a weak similarity if it is surjective and there exists a strictly increasing $f : D(Y) \to D(X)$ such that $d_X = f \circ d_Y \circ F$. It is shown that the weak similarities between geodesic spaces are usual similarities and every weak similarity $F : X \to Y$ is an isometry if X and Y are ultrametric and compact with $D(X) = D(Y)$. Some conditions under which the weak similarities are homeomorphisms or uniform equivalences are also found.

Key words: isometry, similarity, weak similarity, ultrametric, geodesic, semimetric, rigidity of distance set.

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1 Introduction

In the paper we define the notion of weak similarities of semimetric spaces and study some properties of these mappings. Before doing this work we remaind some definitions and introduce related designations.

Let X be a set. A *semimetric* on X is a function $d : X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, such that $d(x, y) = d(y, x)$ and $(d(x, y) = 0) \Leftrightarrow (x = y)$ for all $x, y \in X$. A pair (X, d), where d is a semimetric on X, is called a *semimetric space* (see, for example, [1, p. 7]). A semimetric d is a *metric* if, in addition, the *triangle inequality* $d(x, y) \leqslant d(x, z) + d(z, y)$ holds for all $x, y, z \in X$. A metric is an *ultrametric* if we have the *ultrametric inequality* $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$ instead of the triangle one. We shall denote by **SM**, **M** and **UM** the classes of the nonvoid semimetric spaces, the nonvoid metric spaces and, respectively, the nonvoid ultrametric ones.

Let (X, d_X) and (Y, d_Y) be semimetric spaces. A mapping $\Phi : X \to Y$ is a *similarity* if Φ is bijective and there is a positive number $r = r(\Phi)$, the *ratio* of Φ, such that

$$d_Y(\Phi(x), \Phi(y)) = r d_X(x, y)$$

for all $x, y \in X$ (cf. [7, p. 45]). The *isometries* are similarities with the ratio $r = 1$. The semimetric spaces X and Y are said to be *isometric* if there exists an isometry $F : X \to Y$. We define the *distance set* $D(X)$ of a nonvoid semimetric space (X, d) as

$$D(X) := \{d(x, y) : x, y \in X\}.$$

The following concept seems to be a natural generalization of similarities of semimetric spaces.

Definition 1.1. *Let (X, d_X), $(Y, d_Y) \in \boldsymbol{SM}$. A surjective mapping $\Phi : X \to Y$ is a weak similarity if there is a strictly increasing function $f : D(Y) \to D(X)$ such that the equality*

$$d_X(x, y) = (f \circ d_Y)(\Phi(x), \Phi(y)) \tag{1.1}$$

holds for all $x, y \in X$. Where $f \circ d_Y$ denotes the composition of the functions f and d_Y. Here a function f is said to be a scaling function *of Φ.*

If $\Phi : X \to Y$ is a weak similarity, we write $X \overset{w}{=} Y$, say that X and Y are *weak equivalent* and that the pair (f, Φ) is a *realization* of $X \overset{w}{=} Y$.

It is clear that every similarity is a weak similarity. Moreover a weak similarity $\Phi : X \to Y$ with a scaling function $f : D(Y) \to D(X)$ is a similarity with a ratio r if and only if

$$f(t) = \frac{1}{r} t \tag{1.2}$$

for every $t \in D(Y)$ (see Lemma 3.2 below). It was shown in [5, Theorem 3.6] that if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a bijection such that $f \circ d$ and $f^{-1} \circ d$ are metrics for **every** metric d, then (1.2) holds with some $r > 0$ for every $t \in \mathbb{R}^+$. In the present paper we have found some conditions under which a weak similarity $\Phi : X \to Y$ is a similarity or even an isometry for **given** X and Y.

More precisely:

- Theorem 3.1 shows that every weak similarity $\Phi : X \to Y$ is an isometry if X and Y are ultrametric and compact with $D(X) = D(Y)$;

- Corollary 3.17 of Theorem 3.16 describes the general structural conditions for the set $D(X)$ under which all weak similarities $\Phi : X \to X$ are isometries;

- In Theorem 4.4 we prove that a weak similarity $\Phi : X \to Y$ is a similarity if its scaling function f and the inverse f^{-1} are subadditive in some generalized sense;

- Using Theorem 4.4 we show that every weak similarity $\Phi : X \to Y$ is a similarity if X and Y are geodesic spaces (see Theorem 4.7).

Moreover, in Section 1 we study some common properties of weak similarities and, in Section 2, find conditions under which weak similarities are homeomorphisms (see Proposition 2.1) or uniform equivalences (see Proposition 2.2).

Proposition 1.2. *The relation* $\overset{w}{=}$ *is an equivalence on the class* ***SM***.

Proof. We must show that $\overset{w}{=}$ is reflexive, symmetric and transitive.
Reflexivity. To prove the reflexivity it suffices to take $X = Y$, $f(t) \equiv t$ and $\Phi(x) \equiv x$ in (1.1).
Symmetry. Let (X, d_X), $(Y, d_Y) \in \mathbf{SM}$ and let $X \overset{w}{=} Y$ hold with a realization (f, Φ). Equality (1.1) implies the inequality $(f \circ d_Y)(\Phi(x_1), \Phi(x_2)) > 0$ for every pair of distinct $x_1, x_2 \in X$. Consequently $d_Y(\Phi(x_1), \Phi(x_2)) > 0$ because f is strictly increasing and $f(0) = 0$ (the last equality is also follows from (1.1)). Thus we have

$$(x_1 \neq x_2) \Rightarrow (\Phi(x_1) \neq \Phi(x_2)). \tag{1.3}$$

The surjectivity of Φ and (1.3) imply the existence of the inverse mapping $\Phi^{-1} : Y \to X$. Note also that (1.1) holds for all $x, y \in X$ if and only if

$$d_X = f \circ d_Y \circ (\Phi \otimes \Phi) \tag{1.4}$$

where $\Phi \otimes \Phi$ is a mapping from $X \times X$ to $Y \times Y$ satisfying $\Phi \otimes \Phi(x_1, x_2) = (\Phi(x_1), \Phi(x_2))$ for every $(x_1, x_2) \in X \times X$. Since the left-hand side of (1.4)

is surjective, the function f is also surjective. Consequently f is bijective, so that there is the inverse function $f^{-1} : D(X) \to D(Y)$. Note also that Φ^{-1} is surjective and f^{-1} is strictly increasing. Rewriting (1.4) in the form

$$f^{-1} \circ d_X \circ (\Phi^{-1} \otimes \Phi^{-1}) = d_Y$$

we see that the relation $Y \overset{\mathrm{w}}{=} X$ holds with the realization (f^{-1}, Φ^{-1}). Thus $\overset{\mathrm{w}}{=}$ is symmetric.

Transitivity. Suppose we have $X \overset{\mathrm{w}}{=} Y$ and $Y \overset{\mathrm{w}}{=} Z$ with the corresponding realizations (f, Φ) and (g, Ψ). Since $\Psi \circ \Phi$ is surjective and $f \circ g$ is strictly increasing, $X \overset{\mathrm{w}}{=} Z$ follows from the commutativity of the diagram

$$\begin{tikzcd}
X \times X \arrow[r, "\Phi \otimes \Phi"] \arrow[d, "d_X"'] & Y \times Y \arrow[r, "\Psi \otimes \Psi"] \arrow[d, "d_Y"'] & Z \times Z \arrow[d, "d_Z"'] \\
D(X) & D(Y) \arrow[l, "f"'] & D(Z) \arrow[l, "g"']
\end{tikzcd}$$

where $\Psi \otimes \Psi(y_1, y_2) = (\Psi(y_1), \Psi(y_2))$ for all $y_1, y_2 \in Y$. $\square$

Corollary 1.3. *If $\Phi : X \to Y$ is weak similarity with the scaling function $f : D(Y) \to D(X)$ and, $\Psi : Y \to Z$ is a weak similarity with the scaling function $g : D(Z) \to D(Y)$, then $\Psi \circ \Phi$ is a weak similarity with the scaling function $f \circ g$.*

The proof of Proposition 1.2 gives also the next

Corollary 1.4. *If (f, Φ) is a realization of the equivalence $X \overset{\mathrm{w}}{=} Y$, then f and Φ are bijective.*

The following is closely related to Proposition 2.2 in [2].

Proposition 1.5. *Let $X \in$ **UM**. Then the relation $X \overset{\mathrm{w}}{=} Y$ implies the membership $Y \in$ **UM** for every $Y \in$ **SM**.*

We leave this proposition without any proof as an exercise to the reader.

Remark 1.6. Simple examples show that, in general, the membership $X \in \mathbf{M}$ and the relation $X \overset{\mathrm{w}}{=} Y$ do not imply $Y \in \mathbf{M}$. It can be shown that for an increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ the function $f \circ d$ is a metric for every $(X, d) \in \mathbf{M}$ if and only if f is subadditive, $f(0) = 0$, and $f(x) > 0$ for every $x > 0$ (see Theorem 4.1 in [4]).

Let d and ρ be two semimetrics defined on the same set X. Then d and ρ are said to be *coincreasing* if the equivalence

$$(d(x,y) \leqslant d(z,w)) \Leftrightarrow (\rho(x,y) \leqslant \rho(z,w))$$

holds for all $x, y, z, w \in X$ (cf. Definition 3.1 from [5]). The following proposition is an analogy to Lemma 3.1 from [6].

Proposition 1.7. *Let (X, d_X), $(Y, d_Y) \in \boldsymbol{SM}$ and let $X \xrightarrow{\Phi} Y$ be a bijection. The mapping $(X, d_X) \xrightarrow{\Phi} (Y, d_Y)$ is a weak similarity if and only if there is a semimetric ρ_X on X such that ρ_X and d_X are coincreasing and $(X, \rho_X) \xrightarrow{\Phi} (Y, d_Y)$ is an isometry.*

The simple proof is omitted here. Proposition 1.7 shows, in particular, that the weak similarities are closely connected with the isotone degenerate metric products. See [6] for the exact definitions and some results in this direction.

2 Weak equivalence, homeomorphism and uniform equivalence

Now we present conditions under which weak equivalent metric spaces are homeomorphic. For every $A \subseteq \mathbb{R}^+$ we shall denote by $\mathrm{ac}A$ the set of all accumulation points of A in the space $\mathbb{R}^+$ with the standard topology.

Proposition 2.1. *Let (X, d_X) and (Y, d_Y) belong to $\boldsymbol{M}$ and let $X \stackrel{\mathrm{w}}{=} Y$. Suppose that the equivalence*

$$(0 \in \mathrm{ac}D(X)) \Leftrightarrow (0 \in \mathrm{ac}D(Y)) \tag{2.1}$$

holds. Then X and Y are homeomorphic.

Proof. Assume that 0 is an isolated point for both $D(X)$ and $D(Y)$. Then X and Y are discrete as topological spaces. Let (f, Φ) be a realization of $X \stackrel{\mathrm{w}}{=} Y$. By Corollary 1.4 the mapping $\Phi : X \to Y$ is a bijection. Every bijection between discrete topological spaces is a homeomorphism. Thus X and Y are homeomorphic in the case under consideration.

Consider now the case when $0 \in \mathrm{ac}D(X) \cap \mathrm{ac}D(Y)$. To prove that X and Y are homeomorphic it suffices to show that the weak similarities Φ and Φ^{-1} are continuous. By definition Φ is continuous if the equality

$$\lim_{n\to\infty} d_Y(\Phi(x_0), \Phi(x_n)) = 0 \tag{2.2}$$

holds for every $x_0 \in X$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in X$, with $\lim\limits_{n\to\infty} d_X(x_0, x_n) = 0$. Equality (2.2) can be written in the form

$$\lim_{n\to\infty} f^{-1}(d_X(x_0, x_n)) = 0.$$

Hence to prove (2.2) it is sufficient to show that the scaling function $f^{-1} : D(X) \to D(Y)$ is continuous at the point 0. The last is easy to see. Indeed, since $0 \in \mathrm{ac}D(Y)$, for every $\varepsilon > 0$ there is a point $p \in (0, \varepsilon) \cap D(Y)$. Since f^{-1} is a bijection we can find $r \in (0, \infty) \cap D(X)$ such that $f^{-1}(r) = p$. The increase of f^{-1} implies the inclusion $f^{-1}([0, r)) \subseteq [0, \varepsilon)$. Hence f^{-1} is continuous at 0. The continuity of Φ follows. Similarly we can show that Φ^{-1} is continuous. $\square$

It was shown in the previous proof that the scaling function f^{-1} is continuous at 0 if $0 \in \mathrm{ac}D(Y)$. Since a function defined on a subset of $\mathbb{R}$ is continuous if and only if it is right and left continuous, we can obtain

Proposition 2.2. *Let (X, d_X) and (Y, d_Y) belong to $\boldsymbol{SM}$, let $f : D(Y) \to D(X)$ be an increasing bijection and let $t \in D(Y)$. Then f is continuous at t and f^{-1} is continuous at $f(t)$ if and only if*

$$(t \in \mathrm{ac}([t, \infty) \cap D(Y))) \Leftrightarrow (f(t) \in \mathrm{ac}([f(t), \infty) \cap D(X)))$$

and

$$(t \in \mathrm{ac}([0, t] \cap D(Y))) \Leftrightarrow (f(t) \in \mathrm{ac}([0, f(t)] \cap D(X))).$$

Recall that a uniformly continuous mapping $F : X \to Y$ is a *uniform equivalence* if F is bijective and the inverse function $F^{-1} : Y \to X$ is also uniformly continuous [9, p. 2].

Using Proposition 2.2 and the symmetry of the relation $\overset{\mathrm{w}}{=}$ we obtain

Corollary 2.3. *Let (X, d_X), $(Y, d_Y) \in \boldsymbol{M}$ and let $\Phi : X \to Y$ be a weak similarity. Then Φ is a uniform equivalence if and only if (2.1) holds.*

It is well-known that every uniform equivalence preserves the completeness of metric spaces (see, for example, [12, p. 171]). Consequently Corollary 2.3 implies

Proposition 2.4. *Let* (X, d_X), $(Y, d_Y) \in \boldsymbol{M}$, *and* $X \overset{\mathrm{w}}{=} Y$, *and* $0 \in \mathrm{ac}(D(X)) \cap \mathrm{ac}(D(Y))$. *Suppose that* (X, d_X) *is complete. Then* (Y, d_Y) *is also complete.*

The following example shows that there exist metric spaces which are weak equivalent but not homeomorphic.

Example 2.5. Let $\{r_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be strictly decreasing sequences of positive real numbers such that $\lim\limits_{n\to\infty} r_n = 0$ and $\lim\limits_{n\to\infty} p_n = p > 0$. Let $Y := \{y_0, y_1, ..., y_n, ...\}$ and $X := \{x_0, x_1, ..., x_n, ...\}$ be some families of pairwise distinct points. Define semimetrics d_X and d_Y by the rules

$$\begin{aligned} d_X(x_i, x_j) &:= \begin{cases} 0 & \text{if } i = j \\ r_{i\vee j} & \text{if } i \wedge j = 0 \text{ and } i \vee j > 0 \\ r_{i\wedge j} & \text{if } i \wedge j > 0 \text{ and } i \neq j, \end{cases} \\ d_Y(y_i, y_j) &:= \begin{cases} 0 & \text{if } i = j \\ p_{i\vee j} & \text{if } i \wedge j = 0 \text{ and } i \vee j > 0 \\ p_{i\wedge j} & \text{if } i \wedge j > 0 \text{ and } i \neq j \end{cases} \end{aligned} \tag{2.3}$$

where $i \vee j = \max\{i, j\}$ and $i \wedge j = \min\{i, j\}$. It can be proved directly that (X, d_X), $(Y, d_Y) \in \mathbf{UM}$. The functions Φ and f defined as

$$f(0) := 0, \quad f(p_i) := r_i, \quad \Phi(x_0) := y_0, \quad \Phi(x_i) := y_i, \qquad i = 1, 2, ... \tag{2.4}$$

are bijective and, moreover, f is increasing. It follows from (2.3) and (2.4) that

$$d_X(x_i, x_j) = f(d_Y(\Phi(x_i), \Phi(x_j))$$

for all $x_i, x_j \in X$. Consequently we have $X \overset{\mathrm{w}}{=} Y$ with the realization (f, Φ). It still remains to note that X and Y are not homeomorphic, because X has the limit point x_0 but Y is discrete.

In the next example we consider some ultrametric spaces X and Y such that:

- X and Y are homeomorphic,
- $X \overset{w}{=} Y$ with the realization (f, Φ) for which 0 is not a point of continuity of the scaling function $f : D(Y) \to D(X)$.

Example 2.6. Let $\{r_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be the sequences from the previous example. Let $X := \{x_1^1, ..., x_n^1, ...\} \cup \{x_1^2, ..., x_n^2, ...\}$ and $Y = \{y_1^1, ..., y_n^1, ...\} \cup \{y_1^2, ..., y_n^2, ...\}$ be some families of pairwise distinct points. Define semimetrics d_X and d_Y by the rules

$$\begin{aligned} d_X(x,y) &:= \begin{cases} 0 & \text{if } x = y \\ p_i & \text{if } x = x_i^1 \text{ and } y = x_i^2 \text{ or if } x = x_i^2 \text{ and } y = x_i^1 \\ p_1 & \text{otherwise,} \end{cases} \\ d_Y(x,y) &:= \begin{cases} 0 & \text{if } x = y \\ r_i & \text{if } x = y_i^1 \text{ and } y = y_i^2 \text{ or if } x = y_i^2 \text{ and } y = y_i^1 \\ r_1 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

It can be proved directly that X and Y are countable, ultrametric and discrete. Consequently X and Y are homeomorphic.

Let $\Phi : X \to Y$ and $f : D(Y) \to D(X)$ be the functions such that

$$f(r_i) = p_i \ \text{ and } \ \Phi(x_i^j) = y_i^j, \quad j = 1, 2, \quad i = 1, 2, \dots. \tag{2.6}$$

Then Φ and f are bijective and f is increasing. Equalities (2.5) and (2.6) imply $d_X(x, y) = f(d_Y(\Phi(x), \Phi(y)))$ for all $x, y \in X$. Consequently X and Y are weak equivalent. The point 0 is not a point of continuity of the function $f : D(Y) \to D(X)$ because $0 \in \text{ac}D(Y)$ and $0 \notin \text{ac}D(X)$ and f is strictly increasing.

Remark 2.7. Proposition 2.1 and Corollary 2.3 can be proved also when (X, d_X), $(Y, d_Y) \in \mathbf{SM}$ if we suppose that the distance functions d_X and d_Y are continuous. See [1, p.9] for some basic results related to semimetric spaces with continuous distance functions.

3 Rigidity of distance sets, weak similarities and isometries

It this section we have found some conditions under which the weak similarities become isometries.

Theorem 3.1. *Let* $(X, d_X) \in \boldsymbol{UM}$ *and* $(Y, d_Y) \in \boldsymbol{SM}$*. If* $X \overset{\mathrm{w}}{=} Y$*,* $D(X) = D(Y)$ *and* X *is compact, then* X *and* Y *are isometric.*

The next simple lemma is an original point of our considerations.

Lemma 3.2. *Let* (X, d_X)*,* $(Y, d_Y) \in \boldsymbol{SM}$ *and let* $r > 0$*. If we have* $X \overset{\mathrm{w}}{=} Y$ *with a realization* (f, Φ)*, then the following conditions are equivalent*

(i) The mapping $\Phi : X \to Y$ *is a similarity with a ratio* r*.*

(ii) The function $f : D(Y) \to D(X)$ *satisfies the equality*

$$f(t) = \frac{1}{r}t \tag{3.1}$$

for every $t \in D(Y)$*.*

Proof. If Φ is a similarity with a ratio r, then we have

$$d_Y(\Phi(x), \Phi(y)) = r d_X(x, y) \tag{3.2}$$

for all $x, y \in X$. This equality and (1.1) imply

$$d_X(x, y) = \frac{1}{r} d_Y(\Phi(x), \Phi(y)) = f(d_Y(\Phi(x), \Phi(y)))$$

for all $x, y \in X$. Consequently (3.1) holds for every $t \in D(Y)$, so that (i)⇒(ii) follows.

Analogously (3.1) implies (3.2). Since Φ is a bijection, from (3.2) follows that Φ is a similarity. □

Corollary 3.3. *Let* (X, d_X)*,* $(Y, d_y) \in \boldsymbol{SM}$ *and let* $\Phi : X \to Y$ *be a weak similarity with a scaling function* f*. Then* Φ *is an isometry if and only if* $f(t) = t$ *for every* $t \in D(Y)$*.*

A partially ordered set P is called *rigid* if there is one and only one order preserving bijection $F : P \to P$, (see [8, p. 343]). Of course if P is rigid, then the unique order preserving bijection of P is the identical mapping.

Corollary 3.3 implies the following

Corollary 3.4. *Let $X, Y \in \boldsymbol{SM}$ and let $D := D(X) = D(Y)$. If D is rigid and $X \overset{\mathrm{w}}{=} Y$ with a realization (f, Φ), then the weak similarity $\Phi : X \to Y$ is an isometry.*

To obtain conditions under which D is rigid we recall some notions from the theory of ordered sets.

A total-ordered set $(S, \leqslant)$ is *well-ordered* if every nonempty subset of S has a least element. In this case the relation $\leqslant$ is referred to as a *well-ordering*. Similarly a total order $\leqslant$ on a set S is a *converse well-ordering* if every nonempty subset of S has a greatest element. In what follows we consider a subset D of $\mathbb{R}^+$ together with the standard order $\leqslant_D$ induced from $(\mathbb{R}^+, \leqslant)$.

Lemma 3.5. *Let $(D, \leqslant_D)$ be a nonempty subset of $\mathbb{R}^+$. If $\leqslant_D$ is a well-ordering or a converse well-ordering, then D is rigid.*

Proof. It is well known that every order preserving mapping of a well-ordered set onto itself is the identity mapping (see, for example, [8, p. 4]). Hence D is rigid if $\leqslant_D$ is a well-ordering. Using the duality principle [8, p. 47] we obtain that D is also rigid when $\leqslant_D$ is a converse well-ordering. $\square$

A poset P is said to satisfy the ascending chain condition (ACC) if given arbitrary infinite sequence of elements of P

$$p_1 \leqslant p_2 \leqslant \ldots,$$

then there is $n \in \mathbb{N}$ such that $p_n = p_{n+1} = p_{n+2} \ldots$. It is known that a total-ordered set is a converse well-ordered set if and only if ACC holds.

Lemma 3.6. *Let (X, d_X) be a compact nonvoid ultrametric space and let $D := D(X)$. Then the ordered set $(D, \leqslant_D)$ is a converse well-ordered set.*

Proof. Suppose that ACC does not hold for $(D, \leqslant)$. Then there is an infinite strictly increasing sequence

$$r_1 < r_2 < \dots < r_n < r_{n+1} < \dots \tag{3.3}$$

with $r_n \in D$, $n = 1, 2, ...$ Let us denote by x_n and y_n the points of X such that $d_X(x_n, y_n) = r_n$, $n = 1, 2, \dots$.

Since (X, d_X) is compact, there is a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integer numbers such that the sequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{y_{n_k}\}_{k\in\mathbb{N}}$ are convergent. Write

$$x^* := \lim_{k\to\infty} x_{n_k}, \quad y^* := \lim_{k\to\infty} y_{n_k} \tag{3.4}$$

and $r^* := d_X(x^*, y^*)$. Since the function $d_X : X \times X \to D$ is continuous, we have $r^* = \lim\limits_{k\to\infty} r_{n_k}$. Using (3.3) we see that $r^* > 0$. Relations (3.4) are equivalent to

$$\lim_{k\to\infty} d_X(x^*, x_{n_k}) = \lim_{k\to\infty} d_X(y^*, y_{n_k}) = 0.$$

Consequently there is $k_0 \in \mathbb{N}$ such that

$$d_X(x^*, x_{n_k}) < r^* \text{ and } d_X(y^*, y_{n_k}) < r^* \tag{3.5}$$

for every $k \geqslant k_0$. Considering the triangle $(x^*,\ y^*,\ x_{n_k})$ and using the first inequality from (3.5), we see that the ultrametric inequality implies $d_X(x^*, y^*) = d_X(y^*, x_{n_k})$ (see Figure 1). Similarly, the last equality and the second inequality from (3.5) imply $d_X(y^*, x_{n_k}) = d_X(x_{n_k}, y_{n_k})$. Consequently

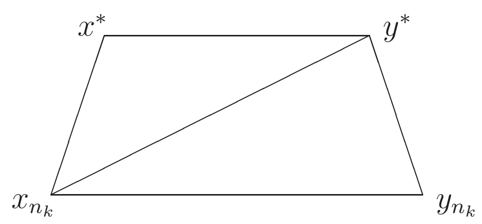

Figure 1: The quadruple $x^*, y^*, x_{n_k}, y_{n_k}$ contains two distinct sides of the maximal length.

if $k > k_0$, then $d(x_{n_k}, y_{n_k}) = d(x^*, y^*)$, contrary to (3.3). Hence the poset $(D, \leqslant_D)$ satisfies ACC, i.e., $(D, \leqslant_D)$ is a converse well-ordered set. $\square$

Proof of Theorem 3.1. Suppose that $X \stackrel{w}{=} Y$, $D := D(X) = D(Y)$ and X is compact. We must show that X and Y are isometric. Let (f, Φ) be a realization of $X \stackrel{w}{=} Y$. By Corollary 3.3, to prove that X and Y are isometric it suffices to show that f is the identical function. Lemma 3.5 implies that $f : D \to D$ is identical if $(D, \leqslant_D)$ is a converse well-ordered set. Since X is compact and ultrametric, it follows from Lemma 3.6 that $\leqslant_D$ is a converse well-ordering. Consequently X and Y are isometric. $\square$

The above given proof also justifies the following

Corollary 3.7. *If $\Phi : X \to X$ is a weak similarity and X is a compact ultrametric space, then Φ is an isometry.*

Let (X, d_X), $(Y, d_Y) \in \mathbf{SM}$. If $X \stackrel{w}{=} Y$, then $D(X)$ and $D(Y)$ must be "isomorphic as ordered sets". Let us recall some related definitions.

Definition 3.8. *[8, p. 45] Let $(P_1, \leqslant)$ and $(P_2, \leqslant)$ be posets. Then $(P_1, \leqslant)$ is said to be* order-isomorphic*, if there exists a bijection $f : P_1 \to P_2$ with the property: For all $a, b \in P_1$ there holds $a \leqslant b \Leftrightarrow f(a) \leqslant f(b)$.*

Definition 3.9. *[8, p. 36] The class of all posets which are order-isomorphic to a given poset $(P, \leqslant)$ is called the* order-type *of $(P, \leqslant)$*

The order-type of a poset P will be defined as $\mathrm{tp}P$. From definitions 1.1, 3.8, and 3.9 we obtain

Proposition 3.10. *Let (X, d_X), $(Y, d_Y) \in \boldsymbol{SM}$. If $X \stackrel{w}{=} Y$, then the equality $\mathrm{tp}(D(X)) = \mathrm{tp}(D(Y))$ holds.*

Corollary 3.11. *Let (X, d_X), $(Y, d_Y) \in \boldsymbol{SM}$ and let $X \stackrel{w}{=} Y$. If $D(X)$ is rigid, then $D(Y)$ is also rigid.*

Proposition 3.12. *Let (X, d_X), $(Y, d_Y) \in \boldsymbol{SM}$ and let $\Phi_i : X \to Y$, $i = 1, 2$, be weak similarities. If $D(X)$ is rigid, then there are isometries $F : X \to X$ and $\Psi : Y \to Y$ such that*

$$\Phi_2 = \Phi_1 \circ F \quad \textit{and} \quad \Phi_2 = \Psi \circ \Phi_1. \tag{3.6}$$

Proof. By Corollary 1.4 the function $F := \Phi_1^{-1} \circ \Phi_2$ is a weak similarity. Suppose that $D(X)$ is rigid. Then Corollary 3.4 implies that F is an isometry on X. Thus

$$\Phi_2 = \Phi_1 \circ F$$

where $F : X \to X$ is an isometry. The first equality from (3.6) is proved. The second can be proved similarly. $\square$

The structure of rigid total-ordered sets was described by A. C. Morel in [10]. To apply his result in our studies we recall the following

Definition 3.13. *Let $(I, \prec)$ be a poset, and let $(P_i, \leqslant_i)$ be posets for $i \in I$ with pairwise disjoint carrier sets P_i. Then we define the* ordered sum $\sum\limits_{i\in I}(P_i, \leqslant_i)$ *of the posets $(P_i, \leqslant_i)$ over the ordered argument $(I, \prec)$ as the poset $(V, \leqslant)$, where $V := \bigcup\limits_{i\in I} P_i$ and where $\leqslant$ is now defined by: For $a, b \in V$ we put $a \leqslant b \Leftrightarrow a$ and b are in the same summand P_i, and there holds $a \leqslant_i b$, or $a \in P_i$ and $b \in P_j$ with $i \neq j$ and $i \prec j$.*

Definition 3.14. *Let $(I, \prec)$ be a poset, and let τ_i, $i \in I$ be order-types. We take a poset $(P_i, \leqslant_i)$ with $\mathrm{tp}(P_i) = \tau_i$ for every $i \in I$, and so that all P_i are pairwise disjoint. Then the sum $\sum\limits_{i\in I}\tau_i$ is the order-type of the ordered sum $\sum\limits_{i\in I}(P_i, \leqslant_i)$. If we have $I = \{1, ..., n\}$ with the standard order $\leqslant$, then we set $\tau_1 + \cdots + \tau_n := \mathrm{tp}\left(\sum\limits_{i\in I}(P_i, \leqslant_i)\right)$.*

In the case when all order-types τ_i are the same we define the product

$$\tau\mathrm{tp}(I) := \mathrm{tp}\left(\sum_{i\in I}(P_i, \leqslant_i)\right)$$

where τ is the common order-type of the posets $(P_i, \leqslant_i)$, $i \in I$.

Lemma 3.15 ([10]). *Let A be a nonempty total-ordered set. The following statements are equivalent.*

(i) There is a nonidentical order preserving bijection $f : A \to A$.

(ii) There are total-ordered sets A_1, A_2, A_3, $A_2 \neq \varnothing$, such that

$$\mathrm{tp}A = \mathrm{tp}A_1 + \mathrm{tp}A_2\mathrm{tp}\mathbb{Z} + \mathrm{tp}A_3$$

where $\mathrm{tp}\mathbb{Z}$ *is the order-type of the set* $\mathbb{Z}$ *of all integer numbers with the standard order.*

Theorem 3.16. *Let* D_1 *and* D_2 *be subset of* $\mathbb{R}^+$ *such that* $0 \in D_1 \cap D_2$. *Then the following statements are equivalent.*

(i) There are sets A_1, A_2, $A_3 \subseteq \mathbb{R}^+$, $A_2 \neq \varnothing$, *such that*

$$\mathrm{tp}(D_1) = \mathrm{tp}(D_2) = \mathrm{tp}(A_1) + \mathrm{tp}(A_2)\mathrm{tp}(\mathbb{Z}) + \mathrm{tp}(A_3). \tag{3.7}$$

(ii) There are (X, d_X), $(Y, d_Y) \in$ ***UM*** *and weak similarities* $\Phi_1 : X \to Y$, $\Phi_2 : X \to Y$ *such that* $D(X) = D_1$ *and* $D(Y) = D_2$ *and* $\Phi_1^{-1} \circ \Phi_2$ *is not an isometry.*

(iii) There are (X, d_X), $(Y, d_Y) \in$ ***SM*** *and weak similarities* $\Phi_1 : X \to Y$, $\Phi_2 : X \to Y$ *such that* $D(X) = D_1$ *and* $D(Y) = D_2$ *and* $\Phi_1^{-1} \circ \Phi_2$ *is not an isometry.*

Proof. **(i)⇒(ii)**. Suppose that A_1, A_2 and A_3 are subsets of $\mathbb{R}^+$ for which (3.7) holds and $A_2 \neq \varnothing$. Then by Lemma 3.15 there exist strictly increasing bijections $\Phi_i : D_1 \to D_2$, $i = 1, 2$, such that $\Phi_1 \neq \Phi_2$. Write $X := D_1$ and $Y := D_2$ and define semimetrics d_X and d_Y by the rules

$$\begin{aligned} d_X(x,y) &:= \begin{cases} 0 & \text{if } x = y,\ x, y \in X \\ \max\{x, y\} & \text{if } x \neq y,\ x, y \in X, \end{cases} \\ d_Y(x,y) &:= \begin{cases} 0 & \text{if } x = y,\ x, y \in Y \\ \max\{x, y\} & \text{if } x \neq y,\ x, y \in Y. \end{cases} \end{aligned} \tag{3.8}$$

Since $\max\{x, y\} \leqslant \max\{x, y, z\} = \max\{\max\{x, z\}, \max\{z, y\}\}$ for all $x, y, z \in \mathbb{R}^+$, the semimetrics d_X and d_Y are ultrametrics. It is clear that $D(X) = D_1$ and $D(Y) = D_2$. Write $f_i := \Phi_i^{-1}$, $i = 1, 2$. Then D_2 is the domain of f_i and D_1 is the range of f_i for $i = 1, 2$. Since Φ_1 and Φ_2 are strictly increasing and bijective, (3.8) implies

$$f_i(d_Y(\Phi_i(x_1), \Phi_i(X_2))) = f_i(\max\{\Phi_i(x_1), \Phi_i(x_2)\})$$

$$= \max\{f_i(\Phi_i(x_1)), f_i(\Phi_i(x_2))\} = \max\{x_1, x_2\} = d_X(x_1, x_2)$$

for $i = 1, 2$ and all distinct $x_1, x_2 \in X$. Moreover we have

$$f_i(d_Y(\Phi_i(x), \Phi_i(x))) = f_i(0) = 0 = d_X(x, x)$$

for $i = 1, 2$ and every $x \in X$. Consequently Φ_1 and Φ_2 are weak similarities. To prove that $\Phi_1^{-1} \circ \Phi_2$ is not an isometry, it is sufficient to find $x, y \in X$ for which

$$d_X(x, y) \neq d_X(\Phi_1^{-1}(\Phi_2(x)), \Phi_1^{-1}(\Phi_2(y))). \tag{3.9}$$

Since the functions Φ_1 and Φ_2 are different and $\Phi_1(0) = \Phi_2(0) = 0$, there is $x_0 \in D_1$ such that $x_0 \neq 0$ and $\Phi_1(x_0) \neq \Phi_2(x_0)$ i.e.,

$$x_0 \neq \Phi_1^{-1}(\Phi_2(x_0)). \tag{3.10}$$

Putting $x = x_0$ and $y = 0$ and using (3.10), (3.8) we obtain

$$d_X(x, y) = \max\{x_0, 0\} = x_0 \neq \Phi_1^{-1}(\Phi_2(x_0))$$

$$= \max\{\Phi_1^{-1}(\Phi_2(x_0)), \Phi_1^{-1}(\Phi_2(0))\} = d_X(\Phi_1^{-1}(\Phi_2(x)), \Phi_1^{-1}(\Phi_2(y)))$$

Relation (3.9) follows.

(ii)⇒(iii). This is trivial.

(iii)⇒(i). Let (iii) hold. Proposition 3.10 implies the equality

$$\operatorname{tp}(D_1) = \operatorname{tp}(D_2). \tag{3.11}$$

We claim that the ordered sets D_1 and D_2 are not rigid. Indeed from (3.11) follows that D_1 is rigid if and only if D_2 is rigid. By Proposition 3.12 if D_1 is rigid, then $\Phi_1^{-1} \circ \Phi_2$ is an isometry (contrary to statement (iii)). Consequently D_1 is not rigid. Lemma 3.15 implies that there are total-ordered sets A_1, A_2, A_3 such that, $A_2 \neq \varnothing$ and

$$\operatorname{tp}(D_1) = \operatorname{tp}(A_1) + \operatorname{tp}(A_2)\operatorname{tp}(\mathbb{Z}) + \operatorname{tp}(A_3).$$

This equality and (3.11) imply (3.7). It is clear that we can take $A_i \subseteq D_1$, $i = 1, 2, 3$. Statement (i) follows. □

Corollary 3.17. *Let $(X, d_X) \in$ **SM**. If not all weak similarities $\Phi : X \to X$ are isometries, then there are A_1, A_2, $A_3 \subseteq D(X)$, $A_2 \neq \varnothing$, such that*

$$\operatorname{tp}(D(X)) = \operatorname{tp}A_1 + \operatorname{tp}A_2\operatorname{tp}\mathbb{Z} + \operatorname{tp}A_3.$$

4 Weak similarities, similarities and geodesics

Recall that a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is subadditive if the inequality

$$f(x + y) \leqslant f(x) + f(y) \tag{4.1}$$

holds for all $x, y \in \mathbb{R}^+$.

Definition 4.1. *([6]) Let A be a subset of $\mathbb{R}^+$. A function $f : A \to \mathbb{R}^+$ is subadditive in the generalized sense if the implication*

$$(x \leqslant \sum_{i=1}^{m} x_i) \Rightarrow (f(x) \leqslant \sum_{i=1}^{m} f(x_i)) \tag{4.2}$$

holds for all $x, x_1, .., x_m \in A$ and every positive integer number $m \geqslant 1$.

Remark 4.2. If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, then (4.1) holds for all $x, y \in \mathbb{R}^+$ if and only if (4.2) is true for all $x, x_1, ..., x_m \in \mathbb{R}^+$ with $m \geqslant 2$. Thus Definition 4.1 is equivalent to the usual definition of subadditivity if $A = \mathbb{R}^+$ and f is increasing.

Lemma 4.3. *([6]) Let A be a nonempty subset of $\mathbb{R}^+$. The following conditions are equivalent for every function $f : A \to \mathbb{R}^+$.*

(i) The function f is subadditive in the generalized sense.

(ii) There is an increasing and subadditive function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is the restriction of Ψ on A.

Theorem 4.4. *Let (X, d_X), $(Y, d_Y) \in$ **SM** and let $\Phi : X \to Y$ be a weak similarity with the scaling function $f : D(Y) \to D(X)$. If f^{-1} and f are subadditive in the generalized sense and $0 \in \operatorname{ac}(D(X)) \cap \operatorname{ac}(D(Y))$, then Φ is a similarity.*

Before proving the theorem we recall the definition of the lower right Dini derivative. Let a real valued function f be defined on a set $A \subseteq \mathbb{R}$ and let $x_0 \in A$. Suppose that $x_0 \in \operatorname{ac}(A \cap (x_0, \infty))$. The lower right Dini derivative D_+ of f at x_0 over set A is defined by

$$D_+ f(x_0) := \liminf_{\substack{x \to x_0 \\ x \in A \cap (x_0, \infty)}} \frac{f(x) - f(x_0)}{x - x_0}.$$

Analogously, the upper right Dini derivative of f at x_0 over set A is defined as

$$D^+f(x_0) := \limsup_{\substack{x\to x_0 \\ x\in A\cap(x_0,\infty)}} \frac{f(x)-f(x_0)}{x-x_0}.$$

Lemma 4.5. *Let $A, B \subseteq \mathbb{R}$ and $f : A \to B$ be strictly increasing and surjective and let $x_0 \in A$, $y_0 := f(x_0)$. If $x_0 \in \operatorname{ac}(A \cap (x_0,\infty))$ and $y_0 \in \operatorname{ac}(B \cap (y_0,\infty))$, then the equality*

$$D^+f^{-1}(y_0) = \frac{1}{D_+f(x_0)} \tag{4.3}$$

holds, where f^{-1} is the inverse function for f and $D^+f^{-1}(y_0) := \{^{\infty}_{0}$ if $D_+f(x_0) = \{^{0}_{\infty}$.

Proof. It follows from the definition of Dini derivatives that

$$\begin{aligned} D^+f^{-1}(x_0) &= \limsup_{\substack{y\to y_0 \\ y\in B\cap(y_0,\infty)}} \frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} \\ &= \left(\liminf_{\substack{y\to y_0 \\ y\in B\cap(y_0,\infty)}} \frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)} \right)^{-1}. \end{aligned}$$

The conditions

$$x_0 \in \operatorname{ac}(A\cap(x_0,\infty)) \text{ and } y_0 \in \operatorname{ac}(B\cap(y_0,\infty))$$

imply that f is right continuous at x_0 and f^{-1} is right continuous at y_0. Consequently we have

$$\liminf_{\substack{y\to y_0 \\ y\in B\cap(y_0,\infty)}} \frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)} = \liminf_{\substack{x\to x_0 \\ x\in A\cap(x_0,\infty)}} \frac{f(x)-f(x_0)}{x-x_0}.$$

Equality (4.3) follows. □

Lemma 4.6. *Let $A \subseteq \mathbb{R}^+$ and let $f : A \to \mathbb{R}^+$ be subadditive in the generalized sense. Suppose that $0 \in A \cap \operatorname{ac}(A)$ and $f(0) = 0$, then the inequality*

$$f(x) \leqslant D_+f(0)x \tag{4.4}$$

holds for every $x \in A$.

Proof. Inequality (4.4) is trivial if $f(x) \equiv 0$ or $D_+f(0) = +\infty$. Hence, without loss of generality, we can assume

$$f(x) \not\equiv 0 \text{ and } D_+f(0) \neq +\infty. \tag{4.5}$$

Since f is subadditive in the generalized sense, (4.5) implies that

$$0 \leqslant D_+f(0) < \infty \tag{4.6}$$

and that the equivalence

$$(f(x) = 0) \Leftrightarrow (x = 0) \tag{4.7}$$

holds for every $x \in A$.

First consider the case when $A = \mathbb{R}^+$. Then, as has been noted in Remark 4.2, f is increasing and subadditive. Every increasing subadditive function satisfying (4.7) is metric preserving (see Theorem 4.1 in [4]), i.e., $f \circ d$ is a metric for every metric space (X, d). As has been shown in Lemma 3.10 [4], a metric preserving function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is Lipschitz if and only if $D_+f(0) < \infty$. Moreover, if this inequality holds, then $D_+f(0)$ is the Lipschitz constant of f. Thus, if $A = \mathbb{R}^+$, then (4.4) holds.

Suppose now that $A \neq \mathbb{R}^+$. By Lemma 4.3 there is an increasing subadditive function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\Psi(x) = f(x) \tag{4.8}$$

for every $x \in A$. Since $A \subseteq \mathbb{R}^+$, the last equality implies the inequality

$$D_+\Psi(0) \leqslant D_+f(0). \tag{4.9}$$

Furthermore we have also

$$(\Psi(x) = 0) \Leftrightarrow (x = 0)$$

for every $x \in \mathbb{R}^+$, because $\Psi(0) = f(0) = 0$ and Ψ is increasing and subadditive. Hence, as has been shown above, the inequality $\Psi(x) \leqslant D_+\Psi(0)x$ holds for every $x \in \mathbb{R}^+$. The last inequality, (4.8) and (4.9) imply (4.4) for every $x \in A$. $\square$

Proof of theorem 4.4. Suppose that f and f^{-1} are subadditive in the generalized sense and

$$a \in \operatorname{ac}(D(X)) \cap \operatorname{ac}(D(y)).$$

Using Lemma 4.6 we obtain the inequality

$$f(y) \leqslant D_+ f(0) y \tag{4.10}$$

for every $y \in D(Y)$ and the inequality

$$f^{-1}(x) \leqslant D_+ f^{-1}(0) x \tag{4.11}$$

for every $x \in D(X)$. Since $D_+ f^{-1}(0) \leqslant D^+ f^{-1}(0)$, inequality (4.11) implies

$$f^{-1}(x) \leqslant D^+ f^{-1}(0) x. \tag{4.12}$$

Note also that the double inequality

$$0 < D_+ f(0) < \infty \tag{4.13}$$

holds. Indeed, if $D_+ f(0) = 0$, then this equality, the inequality $f(y) \geqslant 0$ and (4.10) imply $f(y) \equiv 0$, contrary to bijectivity of f. If $D_+ f(0) = +\infty$, then Lemma 4.5 gives $D^+ f^{-1}(0) = 0$. This equality and (4.12) imply $f^{-1}(x) \equiv 0$, contrary to bijectivity of f^{-1}.

By Lemma 4.5 we have $D^+ f^{-1}(0) = \frac{1}{D_+ f(0)}$. Substituting this equality in (4.12) we obtain

$$f^{-1}(x) \leqslant \frac{1}{D_+ f(0)} x$$

for $x \in D(X)$ or, in the equivalent form,

$$y \leqslant \frac{1}{D_+ f(0)} f(y) \tag{4.14}$$

for $y \in D(Y)$. Inequalities (4.10), (4.13) and (4.14) give the equality $f(y) = D_+ f(0) y$ for every $y \in D(Y)$. Now Lemma 3.2 implies that the weak similarity $\Phi : X \to Y$ is a similarity, as required. $\square$

The geodesic spaces are an important example of spaces for which every weak similarity is a similarity.

We recall the definition of geodesics. Let (X, d) be a metric space. A *geodesic path* in X is a path $\gamma : [a, b] \to X$, $-\infty < a < b < \infty$, such that $d(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1|$ for all $t_1, t_2 \in [a, b]$. If $\gamma(a) = x$ and $\gamma(b) = y$, then we say that γ joins the points x and y. A metric space (X, d) is geodesic if for every two distinct points $x_1, x_2 \in X$ there is a geodesic path in X joining them (see, for example, [11, p. 51, p 58]).

Theorem 4.7. *Let (X, d_X) and (Y, d_Y) be geodesic metric spaces and let $\Phi : X \to Y$ be a weak similarity. Then Φ is a similarity and, if X and Y are bounded with $0 < \operatorname{diam} X \wedge \operatorname{diam} Y$, then the ratio $r(\Phi)$ equals $\frac{\operatorname{diam} Y}{\operatorname{diam} X}$.*

Proof. Let $f : D(Y) \to D(X)$ be the scaling function corresponding Φ. By Theorem 4.4 to prove that Φ is a similarity it is sufficient to show that f and f^{-1} are subadditive in the generalized sense. By definition f is subadditive in the generalized sense if the inequality

$$f(t) \leqslant \sum_{i=1}^{m} f(t_i) \tag{4.15}$$

holds for $t, t_1, t_2, ..., t_m \in D(Y)$ whenever

$$t \leqslant \sum_{i=1}^{m} t_i, \quad m \in \mathbb{N}. \tag{4.16}$$

Let $t, t_1, t_2, ..., t_m \in D(Y)$, $t > 0$, and let (4.16) hold. Write

$$\alpha := \frac{t}{\sum\limits_{i=1}^{m} t_i}.$$

It is clear that $\alpha \leqslant 1$ and $t = \sum\limits_{i=1}^{m} \alpha t_i$. Let $a, b \in Y$ with $d_Y(a, b) = t$ and let $\gamma : [0, t] \to Y$ be a geodesic path joining a and b. Let us define points $y_0, y_1, ..., y_m \in Y$ and $x_0, x_1, ..., x_m \in X$ as

$$y_0 := \gamma(0) = a,\ y_1 := \gamma(\alpha t_1),\ y_2 := \gamma(\alpha t_1 + \alpha t_2), ...,$$

$$y_m := \gamma(\sum_{i=1}^{m} \alpha t_i) = \gamma(t) = b$$

and write $x_i := \Phi^{-1}(y_i)$, $i = 0, ..., m$. The triangle inequality implies

$$d_X(x_0, x_m) \leqslant \sum_{i=0}^{m-1} d_X(x_i, x_{i+1}).$$

Since Φ is a weak similarity with the scaling function f, the last inequality can be written as

$$(f \circ d_Y)(y_0, y_m) \leqslant \sum_{i=0}^{m-1} (f \circ d_Y)(y_i, y_{i+1})$$

or as

$$(f \circ d_Y)(a, b) \leqslant (f \circ d_Y)(\gamma(0), \gamma(\alpha t_1))$$
$$+(f \circ d_Y)(\gamma(\alpha t_1), \gamma(\alpha t_1 + \alpha t_2)) + ... + (f \circ d_Y)(\gamma(\sum_{i=1}^{m-1} \alpha t_i), \gamma(\sum_{i=1}^{m} \alpha t_i)).$$

Since γ is a geodesic joining a and b we have from the previous inequality that

$$f(t) \leqslant f(\alpha t_1) + f(\alpha t_2) + ... + f(\alpha t_m).$$

The last inequality, the increase of f and inequality $\alpha \leqslant 1$ implies (4.15). Consequently f is subadditive in the generalized sense. The generalized subadditivity of f^{-1} can be proved similarly. It still remains to note that the equality $r(\Phi) \operatorname{diam} X = \operatorname{diam} Y$ holds for every similarity $\Phi : X \to Y$. $\square$

Corollary 4.8. *Let (X, d_X) and (Y, d_Y) be bounded geodesic spaces. If* $\operatorname{diam} X = \operatorname{diam} Y$ *and* $X \overset{w}{=} Y$, *then X and Y are isometric.*

To construct an example of compact weak equivalent metric spaces which are not isometric but have the same diameter we shall use the *snow-flake transformation* $d \mapsto d^p$, $p \in (0, 1)$. It is well known d^p is a metric for every metric d and $p \in (0, 1]$ (see, for example, [3, p. 97]).

Example 4.9. Let $X = [0, 1]$, $d_X(x, y) = \sqrt{|x - y|}$ and $Y = [0, 1]$, $d_Y(x, y) = |x - y|$. It is clear that $D(Y) = D(X) = [0, 1]$. The spaces (X, d_X) and (Y, d_Y) are weak equivalent with the realization (f, Φ) where $f(x) = \sqrt{x}$ and $\Phi(x) = x$ for every $x \in [0, 1]$. It is easy to see that (X, d_X) and (Y, d_Y) are compact and $\operatorname{diam} X = \operatorname{diam} Y = 1$.

The space (Y, d_Y) is geodesic. Since there are no rectifiable paths joining 0 and 1 in X, the space (X, d_X) is not geodesic. Hence (X, d_X) and (Y, d_Y) are not isometric.

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